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ON DIVISION RINGS GENERATED BY POLYCYCLIC GROUPS

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ABSTRACT

Let $D = F(G)$ be a division ring generated as a division ring by its central subfield F and the polycyclic-by-finite subgroup G of its multiplicative group, let n be a positive integer and let X be a finitely generated subgroup of $\mathrm{GL}(n, D)$. It is implicit in recent works of A. I. Lichtman that X is residually finite. In fact, much more is true. If $\mathrm{char}\, D = p \neq 0$, then there is a normal subgroup of X of finite index that is residually a finite p-group. If $\mathrm{char}\, D = 0$, then there exists a cofinite set $\pi = \pi(X)$ of rational primes such that for each p in π there is a normal subgroup of X of finite index that is residually a finite p-group.

Let $D = F(G)$ be a division ring generated as a division ring by its central subfield F and the polycyclic-by-finite subgroup G of its multiplicative group D^* and let n be a positive integer. Implicit in the works [5] and [6] of Lichtman is the fact that every finitely generated subgroup of $\mathrm{GL}(n, D)$ is residually finite. Here we prove something sharper.

THEOREM 1. *With D and n as above, let X be any finitely generated subgroup of $\mathrm{GL}(n, D)$ or, more generally, let X be any subgroup of the group of units of a finitely generated subring R of the n by n matrix ring $D^{(n\times n)}$.*

(a) *If $\mathrm{char}\, D = 0$ there exists a cofinite set $\pi = \pi(X)$ of rational primes such that for each $p \in \pi$ there is a normal subgroup of X of finite index that is residually a finite p-group.*

(b) *If $\mathrm{char}\, D = p > 0$ there is a normal subgroup of X of finite index that is residually a finite p-group.*

This theorem directly generalizes a result ([16] 4.7) for linear groups. It is also

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at least superficially related to the main result of Segal's paper [14]. Theorem 1 is an easy consequence of the following.

THEOREM 2. *With D and n as above let R be a finitely generated subring of the matrix ring* $D^{(n\times n)}$. *Then there exists an ideal* $\mathfrak{a}$ *of R of finite index with* $\bigcap_i \mathfrak{a}^i = \{0\}$. *Further, if* $\operatorname{char} D = 0$ *there exists a cofinite set* $\pi = \pi(R)$ *of rational primes such that for each* $p \in \pi$ *we can find* $\mathfrak{a}$ *as above with* $p \in \mathfrak{a}$.

Of course, if $\operatorname{char} D = p > 0$ in Theorem 2, then necessarily $p1_R \in \mathfrak{a}$. There are a number of easy corollaries of Theorem 1.

COROLLARY 1. *Let D, n and X be as in Theorem* 1.

(a) *If* $n = 1$ *or* $\operatorname{char} D = 0$ *then X is torsion-free by finite.*

(b) *If* $\operatorname{char} D = p > 0$ *then X has a normal subgroup of finite index each of whose elements of finite order is a p-element.*

PROOF. Part (b) and the second part of (a) follow at once from Theorem 1. Since D^* contains no non-trivial elements of order char *D*, the first part of (a) follows from (b).

COROLLARY 2. *Let D, n and X be as in Theorem* 1. *If* $\operatorname{char} D = 0$ *then X is centrally eremitic and contains a normal subgroup of finite index with eccentricity* 1. *If* $\operatorname{char} D = p > 0$ *then X is centrally p'-eremitic and contains a normal subgroup of finite index with eccentricity* 1.

This is an immediate consequence of Theorem 1 and paragraph 2.2 of [15]. See [15] or [16] for definitions.

COROLLARY 3 (Lichtman [6] theorem 2). *Let D, n and X be as in Theorem* 1. *If X is also periodic then X is finite.*

PROOF. If $\operatorname{char} D = 0$ then X is finite by Corollary 1(a). Let $\operatorname{char} D = p > 0$. Then X has a normal p-subgroup P of finite index by Corollary 1(b). But P is unipotent and hence nilpotent ([6] theorem 1) and is also a finitely generated p-group. Consequently P is finite and therefore X is too.

Doubtless it is known that in general finitely generated skew linear groups need not be residually finite, but we include a couple of examples at the end of this paper.

THE PROOFS. The standard proof of Higman's zero-divisor theorem ([2] theorem 12) yields the following, the terminology of which we explain below.

(1) *Let* $R = S[G]$ *be a ring,* S *a subring of* R *and* G *a subgroup of the units of* R *normalizing* S *such that* $S \cap G$ *is a subgroup of* G *and* R *is a crossed product of* S *and* $G/(S \cap G)$. *Suppose that* $G/(S \cap G)$ *is locally indicable and that* x *is a non-zero element of* R *each of whose non-zero coefficients in* S *is not a zero-divisor of* S. *Then* x *is not a zero-divisor of* R.

If T is any transversal of $S \cap G$ to G the crossed product condition above amounts to saying that each r in R has a unique representation $r = \Sigma\, tr_t$ where the coefficients r_t lie in S and almost all are zero. A different choice of T multiplies these coefficients by units of S. Thus the hypothesis on x is independent of the choice of T. A group X is locally indicable if each of its finitely generated subgroups has an infinite cyclic image. Note that a poly-$\mathbf{Z}$ group is locally indicable.

(2) *Let* R *be a ring,* J *a subring of* R *such that* R *is finitely generated as right* J*-module and* J_1 *a ring direct summand of* J *that is a right Noetherian ring. If* $a \in R$ *is not a left zero-divisor of* R *then* $aR \cap J_1 \neq \{0\}$.

The conclusion of (2) is also valid if J_1 is commutative (or, more generally, locally right Noetherian) instead of right Noetherian.

PROOF. R is a (not necessarily unital) J_1-module via right multiplication and $R = A \oplus B$ as right J_1-module, where J_1 kills A and B is unital and finitely generated. Then B contains a free J_1-submodule M of finite maximal rank m say ([1] 1.9). Now $M \cong_{J_1} aM$ under the obvious map. If $aR \cap J_1 = \{0\}$ then $aM + J_1$ is a free J_1-submodule of B of rank $m+1$. This contradiction of the choice of m shows that $aR \cap J_1 \neq \{0\}$.

Let G be a group. A *plinth for* G is a G-module that is free of finite rank as $\mathbf{Z}$-module such that A is rationally irreducible for every subgroup of G of finite index (or equivalently such that the connected component $(G\rho)^0$ containing 1 is irreducible over $\mathbf{Q}$, where $\rho : G \to \mathrm{GL}(\operatorname{rank} A, \mathbf{Z})$ is the representation of G determined by a choice of basis of A).

(3) *Let* A *be a plinth for the polycyclic-by-finite group* G. *For* $i = 1, 2, \cdots, r$ *let* k_i *be a locally finite field and let* $J = \bigoplus k_iA$. *Suppose we are given an action of* G *on the ring* J *extending the action on* A *and let* $\nu = \Sigma\, \nu_i \in J$ *where each* $\nu_i \in k_iA \setminus \{0\}$. *Then there exists a maximal ideal* $\mathfrak{m}$ *of* J *with* $\nu^G \cap \mathfrak{m} = \varnothing$.

Hence ν^G denotes the orbit $\{\nu^g : g \in G\}$. Result (3) is a slight generalisation of [9] theorem E.

PROOF. Each k_iA is a domain (e.g. by 1). In particular, G permutes the k_iA. Let $N = \bigcap_i N_G(k_iA)$. Then N is a normal subgroup of G of finite index. Choose a transversal T of N to G. If $t \in T$ then $\nu_i^t \in k_jA$ for some $j = j(t, i)$ and trivially $\nu_i^t \neq 0$. Let ω be the product of all the ν_i^t in k_1A.

By [9] theorem E there is a maximal ideal $\mathfrak{m}_1$ of k_1A with $\omega^N \cap \mathfrak{m}_1 = \emptyset$. Let $\mathfrak{m} = \mathfrak{m}_1 \oplus \Sigma_{i>1} k_iA$. Clearly $\mathfrak{m}$ is a maximal ideal of J. Let $g \in G$. Then $g = th$ for some $t \in T$ and $h \in N$, and $\nu_i^t \in k_1A$ for some unique i. Now ν_i^g divides $\omega^h \notin \mathfrak{m}_1$, so $\nu_i^g \notin \mathfrak{m}_1$ and $\nu^g \notin \mathfrak{m}$ as required.

(4) *Let $R = S[H]$ be a ring, where S is a finite semisimple ring, H a subgroup of the units of R normalizing S, $S \cap H$ a subgroup of H and R a crossed product of S and $H/(S \cap H)$. Let G be a polycyclic-by-finite group of automorphisms of R normalizing S and H, for which $H/(S \cap H)$ is a plinth. Let m be a positive integer and a any non-zero-divisor of R. Then there exists a G-invariant ideal $\mathfrak{a} \neq R$ of R of finite index such that a is a unit modulo $\mathfrak{a}$ and $\mathfrak{a} = \mathrm{rad}_R(\mathfrak{a} \cap \mathrm{rg}\{H^m\})$.*

If X is a subset of a ring R then $\mathrm{rad}_R X$ denotes the intersection of the prime ideals of R containing X ($= R$ if none such exist) and $\mathrm{rg}\{X\}$ denotes the subring of R generated by X (and the identity of R).

PROOF. Clearly $B = H^{|S|!}$ centralizes S and $H' \leqq S \cap H$, which has order dividing $|S|!$ Thus B stabilizes the series $H \supseteq S \cap H \supseteq \langle 1 \rangle$ and $B^{|S|!}$ centralizes H. Set $A = H^l$ where $l = m(|S|!)^3$. Then A is a free abelian subgroup of H of finite index lying in H^m that is normalized by G and central in R. Note that A is also a plinth for G.

The subring k of R generated by its identity element has the form $k = \bigoplus_{i=1}^r k_i$ where each k_i is a finite field. Set $J_i = k_iA = k_i[A] \subseteq R$, so $J = \mathrm{rg}\{A\} = \bigoplus J_i$; we have used here that $S \cap A = \{1\}$. Trivially J is central in R and normalized by G and R is a finitely generated J-module. By (2) there exists $\lambda_i \in aR \cap J_i \setminus \{0\}$. Interchanging right and left there exists also $\mu_i \in Ra \cap J_i \setminus \{0\}$. Let $\nu = \Sigma \lambda_i \mu_i \in J$. By (3) there exists a maximal ideal $\mathfrak{m}$ of J with $\nu^G \cap \mathfrak{m} = \emptyset$. Since J is a finitely generated commutative ring, $J/\mathfrak{m}$ is finite. Consequently $R/\mathfrak{m}R$ is also finite. Trivially, $\mathfrak{m}R = R\mathfrak{m}$.

Let $g \in G$. Then $a^{g^{-1}}R + \mathfrak{m}R \supseteq (\nu^{g^{-1}}J + \mathfrak{m})R = JR = R$. Thus $aR + \mathfrak{m}^g R = R$ and similarly $Ra + \mathfrak{m}^g R = R$. Consequently a is a unit modulo $\mathfrak{m}^g R$ and therefore also modulo $\mathrm{rad}_R(\mathfrak{m}^g)$. The set $\{\mathfrak{m}^g R : g \in G\}$ is finite since $R/\mathfrak{m}R$ is finite and R is finitely generated. Let

$$\mathfrak{a} = \bigcap_{g \in G} \mathrm{rad}_R(\mathfrak{m}^g) = \mathrm{rad}_R\left(\bigcap_{g \in G} \mathfrak{m}^g R\right).$$

Then $R/\mathfrak{a}$ is a finite semisimple ring and each $\mathrm{rad}_R(\mathfrak{m}^g)/\mathfrak{a}$ is a direct sum of simple components of $R/\mathfrak{a}$. Therefore a is a unit modulo $\mathfrak{a}$.

Now $R = \bigoplus_{x\in X} SAx$ where X is any transversal of $(S\cap H)A$ to H. Also S is a direct sum of irreducible k-modules, so SA is a direct sum of cyclic J-modules, each isomorphic to a direct summand of J and one being J itself. If $J = Je \oplus Jf$ where $1 = e + f$ then

$$\bigcap_{g\in G}(\mathfrak{m}^g e) = \left(\bigcap_{g\in G}\mathfrak{m}^g\right)e$$

and therefore

$$\bigcap \mathfrak{m}^g R = \left(\bigcap \mathfrak{m}^g\right)R \subseteq (\mathfrak{a}\cap J)R \subseteq (\mathfrak{a}\cap \mathrm{rg}\{H^m\})R \subseteq \mathfrak{a}.$$

Consequently

$$\mathfrak{a} = \mathrm{rad}_R(\mathfrak{a}\cap \mathrm{rg}\{H^m\}).$$

Finally $R = J\oplus K$ as J-module for some K and so $\mathfrak{m}R \subseteq \mathfrak{m}\oplus K \neq R$. The proof is complete.

For brevity, call a group G *polyplintic* if G has a series $\langle 1\rangle = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_r = G$ of finite length of normal subgroups such that each factor G_i/G_{i-1} is a plinth for G. Note that every subgroup of a polyplintic group of finite index is polyplintic and that every polycyclic-by-finite group has a polyplintic normal subgroup of finite index.

(5) *Let $R = S[G]$ be a ring, where S is a finite semisimple subring of R, G a subgroup of the units of R normalizing S, $S\cap G$ a subgroup of G and R a crossed product of S and the polyplintic group $G/(S\cap G)$. Let P be a normal subgroup of G of finite index and let $a = \Sigma_{t\in T} ta_t \in R\setminus\{0\}$ where T is a transversal of $S\cap G$ to G and the a_t are zero or units of S. Then there exists an ideal $\mathfrak{a}\neq R$ of R of finite index such that a is a unit modulo $\mathfrak{a}$ and $\mathfrak{a} = \mathrm{rad}_R(\mathfrak{a}\cap \mathrm{rg}\{P\})$.*

Proof. We induct on the length of a plinth series for $G/(S\cap G)$. Let $H/(S\cap G)$ be a normal subgroup of $G/(S\cap G)$ such that $H/(S\cap G)$ is a non-trivial plinth for G and G/H is polyplintic. If X is a transversal of H to G then $R = \bigoplus_{x\in X} xS[H]$. Let $a = \Sigma\, xb_x$ where each $b_x \in S[H]$. By (1) each non-zero b_x is a non-zero-divisor of $S[H]$. Consequently so is $b = \Pi_{b_x\neq 0} b_x$ (multiplied in any fixed order). Let $m = (H : H\cap P)$. Then by (4) there is a G-invariant ideal $\mathfrak{b}\subset S[H]$ of finite index such that b is a unit modulo $\mathfrak{b}$ and $\mathfrak{b} = \mathrm{rad}_{S[H]}(\mathfrak{b}\cap \mathrm{rg}\{H\cap P\})$. By an elementary property of finite semisimple rings each non-zero b_x is also a unit modulo $\mathfrak{b}$.

Now $\mathfrak{b}R = R\mathfrak{b} = \bigoplus_{x\in X} x\mathfrak{b}$ since $\mathfrak{b}$ is G-invariant. By induction applied to $R/\mathfrak{b}R$ there exists an ideal $\mathfrak{a} \supseteq \mathfrak{b}$ of R of finite index with $\mathfrak{a} \neq R$ such that a is a unit modulo $\mathfrak{a}$ and

$$\mathfrak{a}/\mathfrak{b}R = \mathrm{rad}_{R/\mathfrak{b}R}(\mathfrak{a}/\mathfrak{b}R \cap \mathrm{rg}\{P \text{ modulo } \mathfrak{b}R\}).$$

Then

$$\begin{aligned}\mathfrak{a} &= \mathrm{rad}_R(\mathfrak{a} \cap (\mathrm{rg}\{P\} + \mathfrak{b}R))\\ &= \mathrm{rad}_R((\mathfrak{a} \cap \mathrm{rg}\{P\}) + \mathfrak{b}R)\\ &= \mathrm{rad}_R((\mathfrak{a} \cap \mathrm{rg}\{P\}) + \mathrm{rad}_{S[H]}(\mathfrak{b} \cap \mathrm{rg}\{H \cap P\}))\\ &= \mathrm{rad}_R(\mathfrak{a} \cap \mathrm{rg}\{P\})\end{aligned}$$

since if $\mathfrak{c}$ is a G-invariant ideal of $S[H]$ of finite index then $(\mathrm{rad}_{S[H]}\mathfrak{c})^s \subseteq \mathfrak{c}$ for some integer s and so

$$((\mathrm{rad}_{S[H]}\mathfrak{c})G)^s \subseteq \mathfrak{c}G \quad \text{and} \quad \mathrm{rad}_{S[H]}\mathfrak{c} \subseteq \mathrm{rad}_R\,\mathfrak{c}.$$

(6) *Let $R = Z[G]$ be a domain, where Z is a central subring of R, G a subgroup of the units of R, $Z \cap G$ a subgroup of G generating Z as a ring and R a crossed product of Z and the polyplintic group $G/(Z \cap G)$. For each prime p let G_p be a normal subgroup of G of finite index containing $Z \cap G$. Let $a \in R \setminus \{0\}$.*

(a) *If* $\mathrm{char}\, R = 0$ *there is a cofinite set π of rational primes such that for each $p \in \pi$ there exists an ideal $\mathfrak{a} \neq R$ of R of finite index such that a is a unit modulo $\mathfrak{a}$ and $p \in \mathfrak{a} = \mathrm{rad}_R(\mathfrak{a} \cap \mathrm{rg}\{G_p\})$.*

(b) *If* $\mathrm{char}\, R = p > 0$ *there exists an ideal $\mathfrak{a} \neq R$ of R of finite index such that a is a unit modulo $\mathfrak{a}$ and $\mathfrak{a} = \mathrm{rad}_R(\mathfrak{a} \cap \mathrm{rg}\{G_p\})$.*

Proof. Let T be a transversal of $Z \cap G$ to G and let $a = \Sigma_T t a_t$ where each $a_t \in Z$. Let $b = \Pi_{a_t \neq 0} a_t$. Now Z is a finitely generated integral domain and hence so is $Z[b^{-1}]$. If $\mathfrak{n}$ is a maximal ideal of $Z[b^{-1}]$ then $\mathfrak{n}$ has finite index and $Z \cap \mathfrak{n}$ is a maximal ideal of Z. If $\mathrm{char}\, R = 0$, set

$$\pi = \{\mathrm{char}(Z[b^{-1}]/\mathfrak{n}) : \mathfrak{n} \text{ as above}\}.$$

Then π is cofinite.

If $\mathrm{char}\, R = 0$ let $p \in \pi$. Otherwise set $p = \mathrm{char}\, R$. The above shows that there is a maximal ideal $\mathfrak{m}$ of Z, necessarily of finite index, with $p \in \mathfrak{m}$ and $b \notin \mathfrak{m}$. It follows that each non-zero a_t is a unit modulo $\mathfrak{m}$. Also $R/\mathfrak{m}R \cong \bigoplus_{t\in T} t(Z/\mathfrak{m})$ and $\mathfrak{m}R$ is an ideal of R since $\mathfrak{m}$ is central. By (5) there is an ideal $\mathfrak{a} \neq R$ of R containing $\mathfrak{m}R$ such that a is a unit modulo $\mathfrak{a}$ and

$$\begin{aligned}\mathfrak{a} &= \mathrm{rad}_R\,(\mathfrak{a} \cap (\mathrm{rg}\{G_p\} + \mathfrak{m}R))\\ &= \mathrm{rad}_R\,((\mathfrak{a} \cap \mathrm{rg}\{G_p\}) + \mathfrak{m})\\ &= \mathrm{rad}_R\,(\mathfrak{a} \cap \mathrm{rg}\{G_p\})\end{aligned}$$

since $Z = \mathrm{rg}\{Z \cap G\} \subseteq \mathrm{rg}\{G_p\}$.

An ideal $\mathfrak{a}$ of a ring R is right weak A–R if for any submodule N of a finitely generated right R-module M there exists an integer m with $N \cap M\mathfrak{a}^m \subseteq N\mathfrak{a}$. There is a similar notion of left weak A–R and weak A–R means left and right weak A–R.

(7) *Let G be a polycyclic group, J a commutative Noetherian ring and $\mathfrak{a}$ an ideal of the group ring $R = JG$ of finite index. Suppose that G is p-nilpotent for every prime p dividing the characteristic of $R/\mathfrak{a}$. Then $\mathfrak{a}$ is weak* A–R.

PROOF. Since R is Noetherian, it suffices to consider a finitely generated (say right) R-module M and an essential submodule N of M killed by $\mathfrak{a}$ and to prove that some power of $\mathfrak{a}$ kills M (see [1] 11.2).

N is a finitely generated module over the finite ring $R/\mathfrak{a}$, so N is finite. By [4] theorem 3 (or alternatively [10]) the split extension $G[M$ is residually finite. Thus there is a submodule K of M of finite index with $K \cap N = \{0\}$. Since N is essential, $K = \{0\}$ and M is finite.

Let $\mathfrak{b} = J \cap \mathfrak{a}$. By the Artin–Rees Lemma (e.g. [7] 11C) there exists an integer $l \geqq 1$ with $N \cap M\mathfrak{b}^l \subseteq N\mathfrak{b} = \{0\}$. Since $\mathfrak{b}$ is central $M\mathfrak{b}^l$ is an R-submodule of M and N is essential. Therefore $M\mathfrak{b}^l = \{0\}$. We now induct on the composition length of M as J-module.

There exists a maximal ideal $\mathfrak{m}$ of J containing $\mathfrak{b}$ with $M\mathfrak{m} < M$. Clearly $N \cap M\mathfrak{m}$ is essential in $M\mathfrak{m}$ (even if $M\mathfrak{m} = \{0\}$) so by induction $M\mathfrak{m}\mathfrak{a}^r = \{0\}$ for some positive integer r. Then $M\mathfrak{a}^r\mathfrak{m} = \{0\}$ and so $M\mathfrak{a}^r$ is a finitely generated $(J/\mathfrak{m})G$-module. Also $J \cap \mathfrak{a} \subseteq \mathfrak{m}$, so G is p-nilpotent for $p = \mathrm{char}\, J/\mathfrak{m}$. By the theorem of [12] every ideal of $(J/\mathfrak{m})G$ is weak A–R so there exists a positive integer s with $N \cap M\mathfrak{a}^r \cap M\mathfrak{a}^{r+s} \subseteq N\mathfrak{a} = \{0\}$. Therefore $M\mathfrak{a}^{r+s} = \{0\}$.

(8) *Let $R = S[G]$ be a ring where S is a subring of R and G is a subgroup of the units of R normalizing S. Suppose P is a normal subgroup of G of finite index with $P \subseteq S$ and let $\mathfrak{a}$ be a G-invariant right* (*resp. left*) *weak* A–R *ideal of S such that $S/\mathfrak{a}$ is right* (*left*) *Noetherian. Then $\mathfrak{b} = \mathrm{rad}_R\, \mathfrak{a}$ is a right* (*left*) *weak* A–R *ideal of R.*

PROOF. We prove the right version. Now $\mathfrak{a}R = \Sigma_{g \in G}\, \mathfrak{a}g$ is an ideal of R and

$\mathfrak{b}/\mathfrak{a}R$ is the radical of $R/\mathfrak{a}R$. Also R is finitely generated as right S-module and therefore $R/\mathfrak{a}R$ is right S-Noetherian and consequently right Noetherian. Thus some power of $\mathfrak{b}$, say $\mathfrak{b}^r$, lies in $\mathfrak{a}R$ ([3] p. 196, theorem 1 and [1] 1.8).

Let M be a finitely generated right R-module and N a submodule of M. Then M is also finitely S-generated so for some positive integer s we have $N \cap M\mathfrak{a}^s \subseteq N\mathfrak{a}$. Then

$$N \cap M\mathfrak{b}^{rs} \subseteq N \cap M(\mathfrak{a}R)^s = N \cap M\mathfrak{a}^s \subseteq N\mathfrak{a} \subseteq N\mathfrak{b}$$

since $(\mathfrak{a}R)^s = (\mathfrak{a}G)^s = G\mathfrak{a}^s$ as $\mathfrak{a}$ is G-invariant. The result follows.

(9) Remark. It is easy to deduce from (7) and (8) that if G is a polycyclic-by-finite group, J a commutative Noetherian ring and $\mathfrak{a}$ an ideal of JG of finite index, then there exists a weak A–R ideal $\mathfrak{b}$ of JG of finite index with $J \cap \mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathfrak{a}$. This is a weak version of [4] theorem 6.

If $\mathfrak{a}$ is an ideal of a ring R let $\mathscr{C}_R(\mathfrak{a})$ denote the set of all elements of R that are not zero-divisors modulo $\mathfrak{a}$.

(10) *Let R be a right Noetherian ring and $\mathfrak{a}$ an ideal of R with $R/\mathfrak{a}$ semisimple. Then:*

(a) $\mathscr{C}_R(\mathfrak{a}) \subseteq \mathscr{C}_R(\mathfrak{a}^i)$ *for each* $i \geqq 1$.

(b) $\mathscr{C}_R(\mathfrak{a})$ *is a right Ore set modulo* $\mathfrak{a}^i$ *for each* $i \geqq 1$.

(c) *If $\mathfrak{a}$ is right weak* A–R *then $\mathscr{C}_R(\mathfrak{a})$ is a right Ore set in R.*

(d) *If $\mathfrak{a}$ is right weak* A–R *and R is also left Noetherian and a domain then in the classical quotient ring RQ^{-1} for $Q = \mathscr{C}_R(\mathfrak{a})$ we have that* $\bigcap_{i=1}^{\infty} (\mathfrak{a}Q^{-1})^i = \{0\}$.

Proof. (a) We induct on i. We may assume that $\mathfrak{a}^{i+1} = \{0\}$. Suppose $xq = 0$ where $x \in R\setminus\{0\}$ and $q \in \mathscr{C}_R(\mathfrak{a})$. By induction $x \in \mathfrak{a}^i$. Now $\mathfrak{a}^i$ is an $R/\mathfrak{a}$-module and $R/\mathfrak{a}$ is semisimple. Thus $\mathfrak{a}^i$ is a direct sum of irreducible $R/\mathfrak{a}$-modules and there exists an irreducible $R/\mathfrak{a}$-submodule V and an element $v \in V\setminus\{0\}$ with $vq = 0$. But V is isomorphic to a submodule of $R/\mathfrak{a}$, so q is a right zero-divisor on $R/\mathfrak{a}$. This contradiction shows that q is not a right zero-divisor of R. In the same way q is not a left zero-divisor either.

(b) Again, we may assume that $\mathfrak{a}^{i+1} = \{0\}$. By (a) we have $\mathscr{C}_R(\mathfrak{a}) \subseteq \mathscr{C}_R(0)$. But since $\mathfrak{a}$ is now the radical of R we have $\mathscr{C}_R(0) \subseteq \mathscr{C}_R(\mathfrak{a})$ and Small's theorem yields that $\mathscr{C}_R(\mathfrak{a})$ is right Ore (see [1] 2.3).

(c) This follows at once from (b) and a lemma of P. F. Smith ([1] 11.9).

(d) Now $\mathfrak{a}Q^{-1}$ is an ideal of RQ^{-1} and therefore $(\mathfrak{a}Q^{-1})^i = \mathfrak{a}^iQ^{-1}$ for each

$i \geqq 1$. Also, if $aq^{-1} = b \in R$ where $a \in \mathfrak{a}^i$ and $q \in Q$, then $bq \in \mathfrak{a}^i$ and (a) yields that $b \in \mathfrak{a}^i$. Consequently $R \cap \mathfrak{a}^i Q^{-1} = \mathfrak{a}^i$ for all $i \geqq 1$. Thus

$$R \cap \bigcap_i (aQ^{-1})^i = \bigcap_i \mathfrak{a}^i = \{0\}$$

also by a result of Smith ([8] 11.2.13). Therefore

$$\bigcap_i (\mathfrak{a}Q^{-1})^i = \left(R \cap \bigcap_i (\mathfrak{a}Q^{-1})^i\right) Q^{-1} = \{0\}$$

as required.

(11) Proof of Theorem 2. If $R \subseteq S$ are rings and $\mathfrak{b}$ is an ideal of S of finite index containing the rational prime p and satisfying $\bigcap \mathfrak{b}^i = \{0\}$ then $\mathfrak{a} = R \cap \mathfrak{b}$ is an ideal of R of finite index containing p and satisfying $\bigcap a^i = \{0\}$. Also $\mathfrak{a}^{(n \times n)}$ is an ideal of the matrix ring $R^{(n \times n)}$ of finite index containing p and satisfying $\bigcap_i (\mathfrak{a}^{(n \times n)})^i = \{0\}$. There exists a finitely generated subring R_1 of D with $R \subseteq R_1^{(n \times n)}$. Thus we may assume that $n = 1$.

Since $F[G]$ is Noetherian, D is the classical quotient ring of $F[G]$ and so $R \subseteq J[G, a^{-1}]$ for some finitely generated subring J of F and some $a \in J[G] \setminus \{0\}$. Enlarge J so that J is still finitely generated, but is also generated by its group of units. Then J is generated by a finitely generated subgroup U of its group of units (actually the whole group of units is finitely generated by [13] théorème 1) and UG is polycyclic-by-finite. Thus replacing G by UG we may assume that J is an image of $\mathbf{Z}$.

Regard G as a subgroup of $\mathrm{GL}(m, \mathbf{Z})$ for some m (e.g. [16] 2.5, 2.3). Then G has a normal subgroup H of finite index with H connected such that H modulo its centre $\zeta_1(H)$ is polyplintic. Since H is connected $\zeta_1(H)$ is the FC-centre of H and H is orbitally sound in Roseblade's terminology (see [11] p. 383). Then by Theorem C1 or [11] we have that $J[H]$ is a crossed product of $Z = J[\zeta_1(H)]$ and $H/\zeta_1(H)$.

Let K denote the quotient field of J in D. Then $K(G)$ has finite dimension d say as left $K(H)$-space and so $J[G, a^{-1}] \subseteq K(G)$ is isomorphic to a subring of $K(H)^{(d \times d)}$. Therefore $J[G, a^{-1}]$ is isomorphic to a subring of $J[H, b^{-1}]^{(d \times d)}$ for some $b \in J[H] \setminus \{0\}$ and we may assume that $H = G$.

For each prime p choose a p-nilpotent normal subgroup G_p of G of finite index containing $\zeta_1(G)$. By (6) there is a prime p and an ideal $\mathfrak{a}$ of $J[G] = Z[G]$ of finite index with $p \in \mathfrak{a}$ such that a is a unit modulo $\mathfrak{a}$ and $\mathfrak{a} = \mathrm{rad}_{J[G]}(\mathfrak{a} \cap J[G_p])$. By (7) the ideal $\mathfrak{a} \cap J[G_p]$ of $J[G_p]$ is weak A–R. Hence $\mathfrak{a}$ is weak A–R by (8). Let $Q = \mathscr{C}_{J[G]}(\mathfrak{a})$. Then by (10) we have that Q is a right divisor set in $J[G]$, $R \leqq J[G, a^{-1}] \subseteq J[G]Q^{-1} = T$ say, and $\mathfrak{a}Q^{-1}$ is an ideal of T of finite index with $p \in \mathfrak{a}Q^{-1}$ and $\bigcap_i (\mathfrak{a}Q^{-1})^i = \{0\}$. If char $D = 0$ then by (6) we can choose p to be any prime with at most a finite number of exceptions. In view of the opening remarks of this proof we have finished.

The following result may be proved similarly to (1) on page 22 of [16].

(12) *If* $\mathfrak{a}$ *is an ideal of finite index in the finitely generated ring* R *then each* $R/\mathfrak{a}^i$ *is also finite.*

(13) Proof of Theorem 1. By Theorem 2 for a suitable prime p we can find an ideal $\mathfrak{a}$ of R of finite index with $p \in \mathfrak{a}$ and $\bigcap_i \mathfrak{a}^i = \{0\}$, and by (12) each $R/\mathfrak{a}^i$ is finite. Regard R as an $R - X$ bimodule via left and right multiplication and set $C_i = C_X(R/\mathfrak{a}^i)$. Then C_1 is a normal subgroup of X of finite index, each C_1/C_{i+1} is a finite p-group and $\bigcap_i C_i = \langle 1 \rangle$, see [16] 4.6.

(14) Examples. We construct examples of 3-*generator soluble subgroups of the multiplicative groups of division rings that are not residually finite.* Our first example is *nilpotent-of-class-two by cyclic.*

Let p be any prime. For each $i \in \mathbf{Z}$ let

$$H_i = \langle x_i, y_i : [x_i, y_i, x_i] = [x_i, y_i, y_i] = 1, [x_i, y_i]^p = 1 \rangle.$$

Let H be the central product of the H_i amalgamating the $[x_i, y_i]$, to z say. It is easy to check that the centre of H is $Z = \langle x_i^p, y_i^p, z : i \in \mathbf{Z} \rangle$. Let g be the automorphism of H defined by $x_i^g = x_{i+1}$, $y_i^g = y_{i+1}$ for each $i \in \mathbf{Z}$ and let G denote the split extension of H by $\langle g \rangle$. Clearly G is 3-generator and nilpotent-of-class-2 by cyclic. Suppose N is a normal subgroup of G of finite index not containing z. Now $\langle z \rangle = H'$, so $H \cap N \subseteq Z$. But H/Z is infinite, contradicting the finiteness of G/N. Consequently, G is not residually finite.

Let F be any field with a primitive p-th root ζ of unity and identify ζ and z. Let $F[H]$ be the corresponding crossed product of F and $H/\langle z \rangle$ with F central. $F[H]$ is locally Noetherian as H is nilpotent, and a domain by (1). By Goldie's theorem ([1] 1.27) $F[H]$ is an Ore domain; let E denote its classical quotient ring. The automorphism g of H determines an automorphism ϕ say of E. Let $D = E((t;\phi))$ be the division ring of formal power series over E in t satisfying $et = te^{\phi}$ for all e in E, see example 1, page 187 of [3]. Then G is isomorphic to the subgroup $\langle t, H \rangle$ of D^*.

Our second example has *derived length 3* like the previous example but is also *torsion-free*. The group we consider is a well-known example of P. Hall.

Let A be the direct product of copies A_i of the rationals, written multiplicatively. Let $i \mapsto p_i$ be a bijection from $\mathbf{Z}$ to the set of *all* primes. Let x be the automorphism of A that permutes the A_i cyclically and let y be the automorphism of A that for each i raises the elements of A_i to their p_i-th powers. Then $B = \langle y^{\langle x \rangle} \rangle$ is abelian and $H = \langle x, y \rangle$ is metabelian. The split extension $G = H[A$ is torsion-free and soluble of derived length 3. Also A is irreducible as H-module, so G is not residually finite.

Let F be any field. The group ring FA is a domain (e.g. by (1)); let K be its quotient field. The action of H on A extends via linearity to an action of H on K. The corresponding skew group ring KB is a domain by (1), and locally Noetherian since B is abelian. Thus KB is an Ore domain; let E be its classical quotient ring. The automorphisms given by x on K and B extend to one ϕ of E. As in the previous example if D is the division ring $E((t;\phi))$ the subgroup $\langle t, AB \rangle$ of D^* is isomorphic to G.

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